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ALMOST PERMUTATIVE VARIETIES OF ASSOCIATIVE ALGEBRAS OVER AN INFINITE FIELD

O. B. Finogenova* UDC 512.552.4

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We give a complete description of almost permutative varieties of algebras over an infinite field in arbitrary characteristic.

In the present paper, we deal with associative algebras over a field. Recall that a *permutation* identity is one of the form

$$x_1x_2\cdots x_n = x_{1\sigma}x_{2\sigma}\cdots x_{n\sigma},$$

where σ is a nontrivial permutation of a set $\{1, 2, \ldots, n\}$.

Varieties or algebras that satisfy such identities are referred to as *permutative*. Permutation identities were brought up for consideration within the framework of semigroup theory in the late 1950s [1]. In ring theory, to our knowledge, these were first treated in [2]. There, it was proved that every permutative variety of algebras over a field of characteristic 0 has the Specht property. Later, that result was generalized to the case of algebras over an arbitrary Noetherian commutative ring with unity [3]. Apart from combinatorial considerations, permutation identities play a noticeable part in research on structural aspects of ring theory [4].

We are interested in the description of permutative varieties in the language of forbidden algebras. A list of such algebras can be created, for instance, by writing out generating algebras for *almost permutative varieties*, i.e., minimal elements in the lattice of all nonpermutative varieties. In fact, by Zorn's lemma, every nonpermutative variety contains an almost permutative variety. Therefore, a variety is permutative iff it lacks any such generating algebra.

In this paper, we give a complete description of almost permutative varieties of algebras over an infinite field in arbitrary characteristic. Note that a characterization of similar varieties can

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Ural Federal University, pr. Lenina 51, Yekaterinburg, 620083 Russia; olgafinogenova@gmail.com.
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also be obtained for algebras over a finite field. In the case of rings, we may limit ourselves to describing almost permutative varieties generated by a finite ring. The above-mentioned results were announced in [5, 6]; their detailed proofs will be presented later on.

In order to formulate our main results, we need identities and the notation for varieties. As usual, for two elements x and y, the commutator $xy - yx$ is denoted by $[x, y]$.

Throughout the paper, p is assumed to be prime or equal to zero. We introduce a one-parameter series $\mathcal{TZ}_p$. A variety $\mathcal{TZ}_0$ is defined by the identities

$$[x, y]u[z, t] = 0, \tag{1}$$

$$[[x, y][z, t], u] = 0, \tag{2}$$

$$[x[y, z]t, u] = 0, \tag{3}$$

$$x[y, z]t = [x, y][z, t] - [x, z][y, t], \tag{4}$$

$$x[y, z]t + t[y, x]z + z[y, t]x = 0, \tag{5}$$

while $\mathcal{TZ}_p$ ($p > 0$), along with (1)-(5), is defined also by

$$x^p u[y, z] = 0, \tag{6}$$

$$[y, z]ux^p = 0, \tag{7}$$

$$[x^p[y, z], u] = 0, \tag{8}$$

$$[[y, z]x^p, u] = 0, \tag{9}$$

$$[xy^p z, u] = 0, \tag{10}$$

$$[x^p y^p, z] = 0, \tag{11}$$

$$xy^p + y^p x = yxy^{p-1}, \tag{12}$$

$$xy^p z - zy^p x = [z, x]y^p + y^p[z, x]. \tag{13}$$

Our goal is not to minimize the list of identities. Therefore, for instance, identity (3) is included, despite its being an obvious consequence of (2) and (4).

Denote by $\mathcal{TD}_p$ ($p > 2$) a variety satisfying the identities

$$[x, y][z, t] = 0, \tag{14}$$

$$[[x, y], z] = 0, \tag{15}$$

$$x^p = 0. \tag{16}$$

We will write $\mathcal{TD}_0$ for a variety specified by (14) and (15). The last variety in the series, $\mathcal{TD}_2$, is given by

$$x^2 y^2 = 0, \tag{17}$$

$$[x^2, y] = 0. \tag{18}$$

THEOREM. A variety of algebras over an infinite field of characteristic $p \geq 0$ is almost permutative if and only if it coincides with $\mathcal{TZ}_p$ or with $\mathcal{TD}_p$.

The varieties in the formulation of the theorem are generated by algebras that have a visual matrix representation.

Let F be a field and U an algebra over F. Put

$$TZ(U) = \left\{ \begin{pmatrix} a & b & c \\ 0 & 0 & d \\ 0 & 0 & a \end{pmatrix} \right\}, \quad TD(U) = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \right\},$$

where a, b, c, d run through U,

$K_{F,n} = \langle k_1, k_2, \dots \mid k_i k_j = k_j k_i,\ k_i^n = 0,\ i, j = 1, 2 \dots \rangle$ for $n > 0$, and

$K_{F,0} = \langle k_1, k_2, \dots \mid k_i k_j = k_j k_i,\ i, j = 1, 2 \dots \rangle$.

PROPOSITION 1. Let F be an infinite field of characteristic $p \geq 0$. The varieties $\mathcal{TZ}_p$ and $\mathcal{TD}_p$ are generated by algebras $TZ(K_{F,p})$ and $TD(K_{F,p})$, respectively.

From the theorem and Proposition 1, we can readily obtain a characterization of permutative varieties in the language of forbidden algebras.

COROLLARY 1. Let F be an infinite field of characteristic $p \geq 0$. A variety of F-algebras is permutative if and only if it does not contain algebras $TZ(K_{F,p})$ and $TD(K_{F,p})$.

1. AUXILIARY RESULTS

In this section, we look into general properties of almost permutative varieties. Let F be a field. A free F-algebra with a countable set X of generators is denoted by $F\langle X \rangle$ and its elements are called *polynomials*. Recall that an ideal of $F\langle X \rangle$ that is closed under endomorphisms is referred to as a T-*ideal*. It is easy to verify that the set of all polynomials $f(\bar{x})$, for which the equality $f(\bar{x}) = 0$ is an identity of some variety $\mathcal{M}$, form a T-ideal. Such is called an *ideal of identities* for $\mathcal{M}$ and is denoted by $T(\mathcal{M})$. Furthermore, a T-ideal generated by a polynomial f is denoted by $T(f)$. A polynomial is said to be *homogeneous* with respect to a variable x if the degrees of all its monomials with respect to x are equal. A polynomial is *polyhomogeneous* if it is homogeneous with respect to all of its variables. Every polynomial uniquely decomposes into a sum of polyhomogeneous polynomials consisting of the maximum possible number of monomials. Such polyhomogeneous polynomials are called *polyhomogeneous components* of a given polynomial. It is well known that for infinite fields, every T-ideal is *polyhomogeneous*. This means that together with any polynomial f, such an ideal will contain all polyhomogeneous components of f.

LEMMA 1 [2]. Every permutative variety satisfies an identity of the form

$$t_1 \cdots t_n [x, y] t_{n+1} \cdots t_s = 0.$$

Until the end of this section, we denote by $\mathcal{V}$ an arbitrary almost permutative variety of algebras over an infinite field.

LEMMA 2. Let $f(\bar{x}) \notin T(\mathcal{V})$. Then, for some n and s, it is true that

$$t_1 \cdots t_n [t_0, t_{s+1}] t_{n+1} \cdots t_s \in T(f) + T(\mathcal{V}).$$

Proof. The ideal $T(f)+T(\mathcal{V})$ is a T-ideal defining a proper and, hence, permutative subvariety of $\mathcal{V}$. For some n and s, therefore, we arrive at the required inclusion

$$t_1 \cdots t_n [t_0, t_{s+1}] t_{n+1} \cdots t_s \in T(f) + T(\mathcal{V}).$$

The lemmas below will essentially facilitate finding identities for almost permutative varieties. Let I be an arbitrary subset of $F\langle X\rangle$ and $f(x, \bar{t})$ a polynomial. We say that f is *linear in a variable x modulo I* if

$$f(x+y, \bar{t}) - f(x, \bar{t}) - f(y, \bar{t}) \in I.$$

LEMMA 3. Let $f(x, \bar{t})$ be a polyhomogeneous polynomial linear in x modulo $T(\mathcal{V})$, and let the degree of f with respect to x be greater than 1. Then $f(x, \bar{t}) \in T(\mathcal{V})$.

Proof. Denote by k the degree of f with respect to x. By hypothesis, $k > 1$. Suppose $f(x, \bar{t}) \notin T(\mathcal{V})$. In view of Lemma 2,

$$x_1 \cdots x_n [y_1, y_2] x_{n+1} \cdots x_s + g \in T(\mathcal{V});$$

here g is a sum of terms like $uf(b, c_1, \ldots)v$, where u, b, c_i, and v are some polynomials. In view of the linearity condition, we may assume that b is a monomial. Therefore, every letter of b occurs in each term of $f(b, c_1, \ldots)$ at least k times. In other words, no monomial in g is multilinear. The property of $T(\mathcal{V})$ being polyhomogeneous implies that $x_1 \cdots x_n [y_1, y_2] x_{n+1} \cdots x_s \in T(\mathcal{V})$. Contradiction.

LEMMA 4. Let f and g be polynomials.

(a) If $x_1 \cdots x_m f(\bar{y}) z_1 \cdots z_k = 0$ is an identity of $\mathcal{V}$, then $f(\bar{y}) = 0$ is also an identity of $\mathcal{V}$.

(b) If $f(h(\bar{t}), \bar{y}) = 0$ is an identity of $\mathcal{V}$ for any $h \in T(xz)$, then $f(x, \bar{y}) = 0$ likewise is an identity of $\mathcal{V}$.

(c) Let $f(\bar{x}) t g(\bar{z}) = 0$ be an identity of $\mathcal{V}$. Then the following conditions are satisfied:

if $f(\bar{x}) \notin T(\mathcal{V})$ then $[x, y] t g(\bar{z}) \in T(\mathcal{V})$;

if $g(\bar{z}) \notin T(\mathcal{V})$ then $f(\bar{x}) t [y, z] \in T(\mathcal{V})$;

if $g(\bar{z}) \notin T(\mathcal{V})$ and $f(\bar{x}) \notin T(\mathcal{V})$ then $[x, u] t [y, z] \in T(\mathcal{V})$.

Proof. (a), (b) Suppose $f(\bar{x}) \notin T(\mathcal{V})$. By Lemma 2, $\mathcal{V}$ has an identity of the form

$$t_1 \cdots t_n [t_0, t_{s+1}] t_{n+1} \cdots t_s = h(\bar{t}), \tag{19}$$

where $h(\bar{t}) \in T(f)$. In case (a) left multiplication by $x_1 \cdots x_m$ and right multiplication by $z_1 \cdots z_k$ will turn the right part of (19) into an identity. The same effect can be gained in case (b) via substitutions $t_i \mapsto x_i z_i$ ($i = 0, 1, \ldots, s+1$). As a result, in either case (19) will be transformed into a permutation identity. Contradiction.

(c) Again we use Lemma 2. For instance, let $f(\bar{x}) \notin T(\mathcal{V})$. Then $\mathcal{V}$ satisfies identity (19). Multiplying (19) on the right by $tg(\bar{z})$ will turn its right part into an identity, while the left part of (19), after applying items (a) and (b), will assume the desired form $[x, y]tg(\bar{z})$. Similarly, if $g(\bar{z}) \notin T(\mathcal{V})$ then $f(\bar{x})t[y, z] = 0$ is an identity of $\mathcal{V}$. But if, simultaneously, $f(\bar{x}) \notin T(\mathcal{V})$, then we obtain $[x, u]t[y, z] \in T(\mathcal{V})$ by taking the commutator $[y, z]$ as $g(\bar{z})$ in the last equality and using the argument above. The lemma is proved.

In view of Lemma 4(c), there are two possibilities for $\mathcal{V}$: either $\mathcal{V}$ possesses an identity $[x, u]t[y, z]$ or identities for $\mathcal{V}$ are not constructed by multiplying extraneous polynomials not in $T(\mathcal{V})$. We show that the last alternative is impossible. To do this, we need a method proposed in [7].

Let $f(\bar{x})$ be a polynomial multilinear in all variables. We make the convention that $f|_{x_i=v}$ denotes a polynomial obtained by substituting in f a (possibly empty) word v for x_i. For example, if $f = xyz + yzx + yxz$ then $f|_{x=1} = 3yz$. We introduce yet another designation. Let $x_i u_1, \ldots, x_i u_k$ all be monomials in f starting with the letter x_i; then we put $f_{x_i} = u_1 + \cdots + u_k$. It is easy to see that f_{x_i} does not depend on x_i, and

$$f(\bar{x}) = \sum_i x_i f_{x_i}(\bar{x}).$$

For instance, for the same polynomial $f = xyz + yzx + yxz$, we have $f_y = zx + xz$.

LEMMA 5 [7]. Let $f(\bar{x}) = 0$ and $h(\bar{y}) = 0$ be multilinear identities for some variety $\mathcal{M}$. Then, for any i, $m = 1, 2, \ldots,$

$$f|_{x_i=1} z h_{y_m}(\bar{y}) = 0$$

is an identity of $\mathcal{M}$.

Proof. We rewrite $h(\bar{y})$ as follows: $h(\bar{y}) = y_m h_{y_m} + \sum_j u_j y_m v_j$, where u_j and v_j are some polynomials. Consider a polynomial such as

$$g = f|_{x_i=1} z h_{y_m} + \sum_j f|_{x_i=u_j} z v_j.$$

It is easy to see that g is a consequence of h. This fact is obvious if f is a monomial, while in the general case, it follows from the linearity of the construction. It remains to observe that the whole sum (except the first term) is a consequence of f. Thus $f|_{x_i=1} z h_{y_m} \in T(\mathcal{M})$.

LEMMA 6. Every almost permutative variety $\mathcal{V}$ satisfies an identity $[x, u]t[y, z] = 0$.

Proof. Assume the contrary. By Lemma 2, $\mathcal{V}$ has an identity of the form $f(\bar{x}) = 0$, where

$$f(\bar{x}) = x_1 \cdots x_n [x_0, x_{m+1}] x_{n+1} \cdots x_m + g(\bar{x}),$$

and $g \in T([x, y][z, t])$. In this instance we may assume the following: first, $f = 0$ is a multilinear identity, and so $m \geq 2$; second, $f = 0$ has the least degree among all identities of this sort in $\mathcal{V}$. Since $f|_{x_m=1}$ and f are of the same form, but the degree of $f|_{x_m=1}$ is smaller than the degree of f, $f|_{x_m=1}$ cannot lie in $T(\mathcal{V})$. Let $h(\bar{y}) = 0$ be a multilinear identity of minimal degree in $\mathcal{V}$. Then

$h_{y_1} \notin T(\mathcal{V})$. By Lemma 5, $f|_{x_m=1} z h_{y_1} = 0$ is an identity of $\mathcal{V}$. If we apply Lemma 4(c) we obtain $[x,y]u[z,t] \in T(\mathcal{V})$. Contradiction.

LEMMA 7. If an almost permutative variety $\mathcal{V}$ satisfies an identity $[[x,y],z] = 0$, then, for any multilinear polynomial $f(\bar{x})$ in $T(\mathcal{V})$ and any i, a polynomial $f|_{x_i=1}$ also lies in $T(\mathcal{V})$.

Proof. Note that for $h(x,y,z) = [[x,y],z]$, we have $h_x = yz$ (see the notation before Lemma 5). According to Lemma 5, together with any multilinear identity $f(\bar{x}) = 0$, $\mathcal{V}$ also satisfies identities $f|_{x_i=1} uyz = 0$ $(i = 1,2,\ldots)$. In view of Lemma 4(a), therefore, $f|_{x_i=1} = 0$ $(i = 1,2,\ldots)$ are identities as well.

2. PROOF OF THE THEOREM: NECESSITY

In this section, we show that every almost permutative variety $\mathcal{V}$ is contained in one of the varieties mentioned in the formulation of the theorem.

PROPOSITION 2. If $[x,u][y,z] \notin T(\mathcal{V})$, then $\mathcal{V}$ is a subvariety of $\mathcal{TZ}_p$.

Proof. By Lemma 6, $\mathcal{V}$ satisfies identity (1). It is easy to see that $\mathcal{V}$ has identity (2) too. Indeed, in view of Lemma 2,

$$x_1 \cdots x_n [t,y] x_{n+1} \cdots x_s \in T([x,y][z,t]) + T(\mathcal{V}).$$

The substitution $t \mapsto [x,u][z,t]$ turns the above inclusion into the following:

$$x_1 \cdots x_n [[x,u][z,t],y] x_{n+1} \cdots x_s \in T([x,y][z,t][u,v]) + T(\mathcal{V}).$$

To obtain identity (2), it remains to observe that $[x,y][z,t][u,v] = 0$ is an identity of $\mathcal{V}$ and discard redundant variables using Lemma 4(a).

Our present goal is to establish the availability of identities (4) and (5).

LEMMA 8. Suppose that under the hypotheses of Proposition 2,

$$g(x_0,\ldots,x_{m+1}) = x_0 f(x_1,\ldots,x_m) x_{m+1}$$

and $f(\bar{x})$ is linear in all variables modulo $T(\mathcal{V})$. In addition, let

$$g(x_0,\ldots,\underbrace{u[x_i,y]}_{i},\ldots,\underbrace{v[x_j,z]}_{j},\ldots,x_{m+1}) = 0$$

be an identity of $\mathcal{V}$ for any i and j $(0 \le i < j \le m+1)$ and for arbitrary (possibly empty) words u and v. Then $f(x_1,\ldots,x_m) = 0$ is an identity of $\mathcal{V}$.

Proof. Assume $f(x_1,\ldots,x_m) \notin T(\mathcal{V})$. By Lemma 4(a), $g(x_0,\ldots,x_{m+1}) \notin T(\mathcal{V})$, and in view of Lemma 2,

$$x_1 \cdots x_n [y_1,y_2] x_{n+1} \cdots x_s \in T(g) + T(\mathcal{V}).$$

It is not hard to verify that under the hypothesis of the lemma, the substitution $y_1 \mapsto t[y_1,z_1]$, $y_2 \mapsto [y_2,z_2]$ turns the above inclusion into the following:

$$x_1 \cdots x_n [t[y_1,z_1],[y_2,z_2]] x_{n+1} \in T(\mathcal{V}).$$

Now it suffices to use an identity $[x,u]t[y,z]=0$ and apply Lemma 4(a). Ultimately we obtain an identity $[y_1,z_1][y_2,z_2]=0$. Contradiction.

We come back to the proof of Prop. 2.

Put $f(x,y,z,t)=x[y,z]t-([x,y][z,t]-[x,z][y,t])$. It is easy to see that this polynomial satisfies the hypotheses of Lemma 8, Obviously, the polynomials $[u,v]f(x,y,z,t)$ and $f(x,y,z,t)[u,v]$ are in $T([x,y]z[u,v])$. Verification that commutators substituted for any two variables turn $f(x,y,z,t)$ into 0 modulo (1) and (2) is routine and so omitted. By virtue of Lemma 8, $f(x,y,z,t)=0$ is an identity of $\mathcal{V}$. Thus $\mathcal{V}$ satisfies identity (4). Recall that (2) and (4) imply (3).

It is easy to verify that the polynomial $x[y,z]t+t[y,x]z+z[y,t]x$ now satisfies all the conditions imposed on f in Lemma 8. In view of this lemma, therefore, $\mathcal{V}$ possesses identity (5).

If the ground field F is a field of characteristic 0, then the proof of the proposition is completed. We turn to the case $p>0$. Notice that all polynomials in (6)-(9) satisfy the hypothesis of Lemma 3. The polynomials each is linear in x modulo $T(\mathcal{V})$ (recall that $\mathcal{V}$ satisfies (1)-(5)) and has degree p with respect to x. Identities (6)-(9) are satisfied in $\mathcal{V}$ by Lemma 3. A similar argument applies to polynomials in (10) and (11), which are linear in y modulo (1)-(9).

Now we embark on identity (12). In view of Lemma 8, $\mathcal{V}$ satisfies

$$s^+_{p+1}(\bar{y})=\sum_{\sigma\in S_{p+1}} y_{1\sigma}\cdots y_{(p+1)\sigma}=0.$$

In fact, by virtue of (1)-(4),

$$\begin{aligned} &s^+_{p+1}(u[y_1,x],v[y_2,z],y_3,\dots,y_{p+1})\\ &=p!u[y_1,x]v[y_2,z]y_3\cdots y_{p+1}+p!v[y_2,z]u[y_1,x]y_3\cdots y_{p+1}=0,\end{aligned}$$

while all the other hypotheses of Lemma 8 are obvious. Consider a polynomial $h(x,y)=x^py+yx^p+\sum_{s=1}^{p-1}x^syx^{p-s}$. It is not hard to verify that

$$h(x+z,y)-h(x,y)-h(z,y)=\sum_{s=1}^{p-1}\frac{1}{(p-s)!s!}s^+_{p+1}(\underbrace{x,\dots,x}_{s\text{ times}},\underbrace{z,\dots,z}_{p-s\text{ times}},y)\in T(\mathcal{V}).$$

If we apply Lemma 8 to $h(x,y)$ we obtain an identity $h(x,y)=0$. Note that (4) gives rise to an identity $x[x,y]x=0$, which can be used to write the identity proved immediately above in the form $x^py+yx^p=xyx^{p-1}$, i.e., of identity (12).

Finally, we show that $\mathcal{V}$ satisfies (13). To do this, we write the following chain of equalities:

$$\begin{aligned} t[x_2,x_1]y^p+ty^p[x_2,x_1] &\overset{(7)}{=}(-x_1[t,x_2]-x_2[x_1,t])y^p+ty^p[x_2,x_1]\\ &\overset{(11)}{=}-[t,x_2]y^px_1-[x_1,t]y^px_2+ty^p[x_2,x_1]\\ &\overset{(10)}{=}tx_1y^px_2-tx_2y^px_1+ty^px_1x_2-ty^px_2x_1\end{aligned}$$

$$+ty^p[x_2,x_1]$$
$$= tx_1y^px_2 - tx_2y^px_1.$$

Thus $\mathcal{V}$ satisfies

$$t([x_2,x_1]y^p + y^p[x_2,x_1] - x_1y^px_2 + x_2y^px_1) = 0.$$

If we remove the variable t from the above identity in accordance with Lemma 4(a) we arrive at (13).

Thus $\mathcal{V}$ is contained in $\mathcal{TZ}_p$. The proposition is proved.

PROPOSITION 3. Let $\mathcal{V}$ satisfy an identity $[x,y][z,t]=0$. Then $\mathcal{V}$ is contained in $\mathcal{TD}_p$.

Proof. First we show that $[x,[y,z]]=0$ is an identity of $\mathcal{V}$. Assume the contrary. Then, in view of Lemma 2, our variety contains an identity of the form

$$f(\bar{t}) = t_1\cdots t_n[t_{n+1},t_{n+2}]t_{n+3}\cdots t_s + g(\bar{t}) = 0,$$

and $g(\bar{t}) \in T([[x,y],z])$. Suppose that $f(\bar{t})=0$ depends on the minimum possible number of variables among identities of this form. Since $T(\mathcal{V})$ is polyhomogeneous, we may assume that $f(\bar{t})$ is a polynomial of degree 1 with respect to all variables. Clearly, $s\geq 3$, i.e., there is at least one letter to the left or right of the commutator $[t_{n+1},t_{n+2}]$. Let it be on the left. Then

$$f(\bar{t}) = t_1a(t_2,\ldots,t_s) + b(\bar{t}) + c(t_2,\ldots,t_s)t_1,$$

where $a - t_2\cdots t_n[t_{n+1},t_{n+2}]t_{n+3}\cdots t_s \in T([[x,y],z])$, $b,c\in T([[x,y],z])$, and the variable t_1 in b occurs only in the commutators. Therefore,

$$b(zt_1,\ldots,t_s) - (zb(\bar{t}) + b(z,t_2,\ldots,t_s)t_1) \in T([x,y][u,v]).$$

Consider a polynomial such as

$$h(z,\bar{t}) = zf(\bar{t}) + f(z,t_2,\ldots,t_s)t_1 - f(zt_1,\ldots,t_s)$$
$$\overset{(14)}{=} z(a(t_2,\ldots)+c(t_2,\ldots))t_1.$$

Obviously, $h(z,\bar{t})=0$ is an identity of $\mathcal{V}$. Consequently, $a(t_2,\ldots)+c(t_2,\ldots)=0$ is also an identity in view of Lemma 4(a). The last identity has the same form as $f=0$, but a smaller number of variables, the latter being a contradiction with the choice of f. Thus $[[x,y],z]\in T(\mathcal{V})$, proving the proposition for the case where $\operatorname{char} F = 0$.

Now let $\operatorname{char} F = p > 0$.

LEMMA 9. Suppose that a complete linearization $\tilde{f}(\bar{z})$ with respect to all variables of a polynomial f depends on at least three variables, and moreover, $\tilde{f}|_{z_k=1}\in T(\mathcal{V})$ for any k. Then $f=0$ is an identity of $\mathcal{V}$ under the hypotheses of Prop. 3.

Proof. Suppose $f \notin T(\mathcal{V})$. By Lemma 2, $\mathcal{V}$ satisfies a multilinear identity of the form

$$[t_1, t_2]t_3 \cdots t_n = \sum a(\bar{t})\tilde{f}(b_1, b_2, \ldots)c(\bar{t}).$$

This, in view of Lemma 7, remains to be an identity after the substitution

$$t_3 \mapsto 1, \ldots, t_n \mapsto 1.$$

Since $\tilde{f}(\bar{z})$ depends on at least three variables, this substitution will turn the right part into a polynomial in $T(\mathcal{V})$. As a result, we face an identity $[t_1, t_2] = 0$. Contradiction. The lemma is proved.

In order to complete the proof of Proposition 3, as f in the hypothesis of Lemma 9 we need only take a polynomial x^p, if $\mathrm{char} F = p > 2$, and polynomials x^2y^2 and $[x^2, y]$ if $\mathrm{char} F = 2$. Then

$$\tilde{f} = \sum_{\sigma \in S_p} z_{1\sigma} \cdots z_{p\sigma}$$

in the former case, and $\tilde{f} = [z_1, z_2][z_3, z_4]$ or $\tilde{f} = [[x, y], z]$ in the latter case. Obviously, these polynomials satisfy all the necessary conditions.

Thus if $\mathrm{char} F = p > 2$, then identities (14)-(16) are satisfied in $\mathcal{V}$. If, however, $\mathrm{char} F = 2$, then $\mathcal{V}$ satisfies (17) and (18).

3. PROOF OF THE THEOREM: SUFFICIENCY

In this section, we show that the varieties given in the hypothesis of the theorem are almost permutative. We need to verify that, first, the varieties in the hypothesis are all nonpermutative, and second, that their proper subvarieties are all permutative.

LEMMA 10. The varieties $\mathcal{TZ}_p$ and $\mathcal{TD}_p$ contain algebras $TZ(K_{F,p})$ and $TD(K_{F,p})$, respectively. In this case the algebras mentioned are nonpermutative.

The **proof** is an easy exercise and so omitted.

It remains to verify the following:

PROPOSITION 4. All proper subvarieties of $\mathcal{TZ}_p$ and $\mathcal{TD}_p$ are permutative for any $p \geq 0$.

Proof. Let $\mathcal{M}$ be one of the varieties $\mathcal{TZ}_p$ or $\mathcal{TD}_p$. By Lemma 10, $\mathcal{M}$ is nonpermutative. Therefore, $\mathcal{M}$ contains an almost permutative subvariety $\mathcal{V}$. We argue to state that this subvariety cannot be proper.

Suppose the contrary. Then $T(\mathcal{V}) \setminus T(\mathcal{M})$ contains a polynomial $f(\bar{x})$. We show that either $f(\bar{x}) \in T(\mathcal{M})$ or $f = 0$ implies a permutation identity for $\mathcal{V}$.

First let $\mathcal{M} = \mathcal{TD}_p$. We may assume that

$$f(\bar{x}) = \alpha x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n} + \sum [x_i, x_j] h_i(\bar{x}),$$

with $k_i < p$ for all $i = 1, \ldots, n$.

If $\alpha \neq 0$, then we multiply $f(\bar{x})$ by $[y,z]$ and then linearize the resulting polynomial. Such a transformation modulo the ideal $T(\mathcal{TD}_p)$ will give rise to an identity of the form

$$\alpha[y,z]t_1 \cdots t_{k_1+\cdots+k_n} = 0.$$

Contradiction.

Let $\alpha = 0$. If $p = 0$ then we can think of the identity $f = 0$ as being multilinear. In view of Lemma 7, $f = 0$ remains an identity after substituting 1's for some of the variables. From $f = 0$, we can now derive an identity $[x,y] = 0$. Let $p > 0$. Multiplying $f(\bar{x})$ by the respective powers of the variables x_k will produce, modulo $x^p = 0$ $(p > 2)$ and $[y,z]x^2 = 0$ $(p = 2)$, an identity of the form $[x_1,x_2]x_1^{p-1}x_2^{p-1}\cdots x_n^{p-1} = 0$. By virtue of Lemma 4(c), $\mathcal{V}$ satisfies $[x_1,x_2]x_1^{p-1}x_2^{p-1} = 0$. Let $g(x_1,t_1,t_2,\ldots,t_p) = 0$ be a complete linearization of the last identity with respect to x_2. Then it is easy to see that

$$\begin{aligned} g(x_1, zt_1, t_2, \ldots, t_p) &- g(x_1, t_1, t_2, \ldots, t_p)z \\ &\overset{(14)}{=} [x_1, zt_1]x_1^{p-1}t_2\cdots t_p - [x_1,t_1]x_1^{p-1}t_2\cdots t_p z \\ &\overset{(14),(15)}{=} [x_1,z]x_1^{p-1}t_1\cdots t_p. \end{aligned}$$

This, together with Lemma 4(a), implies $[x_1,z]x_1^{p-1} = 0$. If we repeat the previous step we obtain $[y,z]t_1\cdots t_p = 0$. Contradiction.

Now let $\mathcal{M} = \mathcal{TZ}_p$. First we argue to state several facts about $\mathcal{TZ}_p$.

LEMMA 11. The variety $\mathcal{TZ}_p$ satisfies the following identities:

$$x[x,y]x = 0, \tag{20}$$

$$x^{p+1} = 0 \text{ if } p > 0, \tag{21}$$

$$y^p x z^p = 0 \text{ if } p > 0. \tag{22}$$

Proof. Identity (20) derives from (4) if we take $z = t = x$, while (21) follows from (12) for $y = x$. Identity (22) can be obtained by multiplying (12) on the right by z^p, applying (7) to the right part, and cancelling equal monomials xy^pz^p.

LEMMA 12. Let $p > 0$ and $h(x,\bar{y})$ be a polynomial of degree p with respect to x. Then there exists a polynomial $g(x,\bar{y})$ of degree 1 with respect to x such that $h(x,\bar{y}) = g(x^p,\bar{y})$ is an identity of $\mathcal{TZ}_p$.

Proof. It suffices to prove the above statement for a monomial h. Let $h = u_0x^{k_1}u_1x^{k_2}u_2\cdots x^{k_s}u_s$, where u_i are nonempty words over an alphabet $y_1,y_2,\ldots$, while $k_1+\cdots+k_s = p$. We use induction on s. For $s = 1$, the statement is trivial. Let $s \geq 2$. Then

$$\begin{aligned} h &= u_0x^{k_1}[u_1,x^{k_2}]u_2\cdots x^{k_s}u_s + u_0x^{k_1+k_2}u_1u_2\cdots x^{k_s}u_s \\ &\overset{(3)}{=} u_0x^{p-k_2}[u_1,x^{k_2}]u_2\cdots u_s + u_0x^{k_1+k_2}u_1u_2\cdots x^{k_s}u_s \end{aligned}$$

$$= u_0 x^{p-1}[u_1, x]u_2 \cdots u_s + u_0 x^{p-k_2}[u_1, x^{k_2-1}]xu_2 \cdots u_s$$
$$+ u_0 x^{k_1+k_2} u_1 u_2 \cdots x^{k_s} u_s.$$

In the first term of the resulting sum, all x are collected together modulo (12) and (20): i.e,

$$x^{p-1}[u_1, x] = x^{p-1}u_1 x - x^p u_1 \overset{(20)}{=} xu_1 x^{p-1} - x^p u_1 \overset{(12)}{=} u_1 x^p + x^p u_1 - x^p u_1 = u_1 x^p.$$

The second term can be discarded, for it is itself an identity in view of (20). In the third term, all x are collected in x^p by the inductive assumption. Lemma 12 is proved.

LEMMA 13. A subvariety of $\mathcal{TZ}_p$ is permutative if it satisfies one of the identities $[x, y][z, t] = 0$, $y^p xy = 0$, or $x^p y^p + \alpha y^p x^p = 0$ (for $p > 0$).

Proof. By virtue of identity (4), $[x, y][z, t] = 0$ implies a permutation identity $x[y, z]t = 0$.

Suppose that $p > 0$ and that $y^p xy = 0$ is satisfied. In view of (21), it can be written in the form $y^p[x, y] = 0$. Linearizing this identity yields

$$\sum_{i=1}^{p}(p-1)!y_1 \cdots \hat{y}_i \cdots \hat{y}_j \cdots y_{p+1} y_i [x, y_j] = 0$$

($\hat{x}_i$ means that a variable x_i does not occur in a respective product). The substitution $y_1 \mapsto [y_1, z_1]$, $y_2 \mapsto [y_2, z_2]u$ leads, in view of (1), to an identity $y_3 \cdots y_{p+1}[y_1, z_1][y_2, z_2]ux = 0$. To obtain $[y_1, z_1][y_2, z_2] \in T(\mathcal{V})$, it remains to use (4).

Lastly, let $x^p y^p + \alpha y^p x^p = 0$ be satisfied. Linearizing this identity gives

$$\sum_{i=1}^{p}(p-1)!x^p y_i y_1 \cdots \hat{y}_i \cdots y_p + \alpha \sum_{i=1}^{p}(p-1)!y_1 \cdots \hat{y}_i \cdots y_p y_i x^p = 0.$$

Substituting $y_1 \mapsto [y_1, x]u$, using (7), and applying Lemma 4(a) will turn the above identity into $x^p y_1 x = 0$, which implies permutativity, as shown above.

We come back to the proof of Prop. 4. Recall that $\mathcal{V}$ is an almost permutative subvariety of $\mathcal{TZ}_p$, and

$$f(\bar{x}) \in T(\mathcal{V}) \setminus T(\mathcal{TZ}_p).$$

We may assume that for any i, the degree of a polynomial $f(\bar{x})$ with respect to x_i is equal to 1, p, or $p + 1$. Indeed, in the case of zero characteristic, we may deal with multilinear identities only. Let $p > 0$. By Lemma 12, letters that occur in a monomial p times can be collected together. Therefore, (6), (7), (20), and (21) transform into an identity of $\mathcal{TZ}_p$ every monomial of degree not less than $p + 2$ with respect to some variable. In addition, if $f(\bar{x})$ has degree k with respect to x_1, and $1 < k < p$, then we can linearize $f(\bar{x})$ with respect to x_1. A polynomial $g(y, x_1, x_2, \ldots) = f(x_1 + y, x_2, \ldots) - f(x_1, x_2, \ldots) - f(y, x_2, \ldots)$ cannot be contained in $T(\mathcal{TZ}_p)$. In fact, otherwise, for any λ in the ground field,

$$g(\lambda x_1, x_1, x_2, \ldots) = (\lambda + 1)^k f(\bar{x}) - f(\bar{x}) - \lambda^k f(\bar{x})$$

$$= ((\lambda+1)^k - \lambda^k - 1)f(\bar{x}) \in T(\mathcal{TZ}_p).$$

Since $1 < k < p$, the field contains an element λ such that

$$(\lambda+1)^k - \lambda^k - 1 \neq 0.$$

Hence $f(\bar{x}) \in T(\mathcal{TZ}_p)$. We have arrived at a contradiction showing that we can effect a complete linearization of f with respect to variables whose degree is less than p.

Thus the degree of f with respect to any of its variables is equal to 1, p, or $p+1$.

Suppose first that f has at least one variable whose degree is not 1. The case where two or more variables occur in $f(\bar{x})$ in degrees not less than p will rather readily lead us to a contradiction. Due to Lemma 12 and identities (11) and (12), any such polynomial can be written in the form $\alpha x^p y^p x_1 \cdots x_n + \beta y^p x^p x_1 \cdots x_n$. According to Lemma 4(a), $\mathcal{V}$ will then satisfy an identity $\alpha x^p y^p + \beta y^p x^p = 0$. By Lemma 13, therefore, $\alpha = \beta = 0$. This means that $f \in T(\mathcal{TZ}_p)$. Contradiction.

Consider a more complicated case where exactly one variable occurs in each monomial of f at least p times. Then this polynomial, in view of Lemma 12 and identity (10), is rewritten as follows:

$$\begin{aligned}\sum_{i<j}^{n} \beta_{ij} x_i y^p x_j x_1 \cdots \hat{x}_i \cdots \hat{x}_j \cdots x_n + \sum_{i=1}^{n} \gamma_i y^p x_i x_1 \cdots \hat{x}_i \cdots x_n \\ + \sum_{i=1}^{n} \delta_i x_1 \cdots \hat{x}_i \cdots x_n x_i y^p + \lambda y^p x_1 \cdots x_n y.\end{aligned} \tag{23}$$

If $n = 1$, then all β_{ij} are equal to 0. Note also that if $n > 2$, then

$$x_i y^p x_j x_2 = x_i y^p x_2 x_j + x_i y^p [x_j, x_2] \overset{(8)}{=} x_i y^p x_2 x_j + y^p [x_j, x_2] x_i.$$

If we apply (10) and (9) we obtain

$$x_i y^p x_2 x_1 = x_1 y^p x_2 x_i + x_2 [x_1, x_i] y^p.$$

In view of this, we may assume that all β_{ij} in (23) (except maybe β_{12}) are equal to 0. We claim that β_{12}, γ_i, and δ_i equal 0. The substitution $x_k \mapsto [z,t]u$ in (23) leads to an identity $\gamma_k y^p [z,t] u x_1 \cdots \hat{x}_k \cdots x_n = 0$, for $k \neq 2$, and to an identity $(\beta_{12} + \gamma_2) y^p [z,t] u x_1 x_3 \cdots x_n = 0$ for $k = 2$. It is not hard to see that $\gamma_k = 0$ ($k \neq 2$) and $\gamma_2 = -\beta_{12}$. Otherwise, by Lemma 4(a), $\mathcal{V}$ satisfies an identity $y^p[z,t] = 0$; hence it satisfies $y^p z y = 0$, which is impossible in view of Lemma 13. A similar substitution $x_k \mapsto u[z,t]$ ($k = 1, 2, \ldots$) gives rise to equalities $\delta_k = 0$ ($k \neq 1$) and $\delta_1 = -\beta_{12}$. Note that for $n = 1$, all γ_k and δ_k are equal to 0. For $n > 1$, the monomial in (23) modulo $T(\mathcal{TZ}_p)$ is rewritten as follows:

$$f(y, \bar{x}) = \beta_{12}(x_1 y^p x_2 x_3 \cdots x_n - y^p x_2 x_1 x_3 \cdots x_n - x_2 x_3 \cdots x_n x_1 y^p) + \lambda y^p x_1 \cdots x_n y.$$

It is easy to verify that $\beta_{12} = 0$. Otherwise, if we multiply f on the right by $[z,y]$ and use Lemma 4(a) we obtain an identity $y^p[z,y] = y^p z y = 0$ satisfied in $\mathcal{V}$, which implies permutativity in view of Lemma 13. Therefore, we may assume that γ_i, δ_i, and β_{ij} are all equal to 0. The remaining case $\lambda \neq 0$, due to Lemmas 4(b) and 13, leads us to a contradiction with the nonpermutativity of $\mathcal{V}$. Thus if f is not multilinear then $f \in T(\mathcal{T}\mathcal{Z}_p)$.

Now let $f(\bar{x})$ be a multilinear polynomial, and let it have the least number of variables among all such polynomials in $T(\mathcal{V})\backslash T(\mathcal{T}\mathcal{Z}_p)$. Notice that $f(\bar{x})$ must necessarily be the sum of commutator monomials. Otherwise, it is a simple matter to obtain a permutation identity by multiplying $f(\bar{x})$ on the left by $[y, z]u$.

Variables in commutator monomials can be rearranged using the obvious equality

$$[x, y]z + [y, z]x + [z, x]y = x[y, z] + y[z, x] + z[x, y]. \tag{24}$$

By virtue of this fact, we may include x_1 in the commutator and write

$$f(\bar{x}) = \sum_{i=2}^{n} \alpha_i [x_1, x_i] x_2 \cdots \hat{x}_i \cdots x_n + g,$$

where $g \in T(y[z, t])$. Let $\alpha_i \neq 0$ for some i. Multiplying $f(\bar{x})$ on the left by $[u, v]$, making the substitution $x_i \mapsto [x_i, y]$, and applying (1), we obtain an identity of $\mathcal{V}$ of the form $\alpha_i[u, v][x_i, y]x_1x_2 \cdots x_n = 0$. This, by Lemmas 4(a) and (13), implies permutativity, if $\alpha_i \neq 0$. Hence $\alpha_i = 0$ for all i.

In view of (4) and (24), we may assume that

$$f(\bar{x}) = \sum_{i=2}^{n} \beta_i x_2 \cdots \hat{x}_i \cdots x_n [x_1, x_i] + g,$$

where $g \in T([x, y][z, t])$. Now if we multiply $f(\bar{x})$ on the right by $[u, v]$ and make the same substitution we see that $\beta_i = 0$ for all i. Consequently, $f \in T([x, y][z, t])$.

We rewrite f so that x_1 and x_2 will be contained in commutators. This can be done since $\mathcal{V}$ satisfies (2) and the following identities:

$$\begin{aligned}
[a, b][x_j, x_i]x_1 &\overset{(24),(1)}{=} -[a, b][x_i, x_1]x_j - [a, b][x_1, x_i]x_j,\\
[x_1, x_i][a, b]x_2 &\overset{(24),(1)}{=} -[x_1, x_i][b, x_2]a - [x_1, x_i][x_2, a]b,\\
x_2[a, b][x_1, x_i] &\overset{(24),(1)}{=} -a[b, x_2][x_1, x_i] - b[x_2, a][x_1, x_i].
\end{aligned}$$

Furthermore, by virtue of (24), we may assume that for $n \geq 5$, x_1 and x_2 are not both contained in one commutator together. For $n = 4$, (4) and (5) imply

$$\begin{aligned}
[x_1, x_2][x_3, x_4] &= [x_1, x_3][x_2, x_4] - [x_4, x_2][x_1, x_3] + [x_4, x_1][x_2, x_3]\\
&\quad - [x_3, x_2][x_4, x_1] + [x_3, x_4][x_2, x_1].
\end{aligned}$$

For $n = 4$, we therefore assume that x_1 and x_2 both occur together only in the right commutator.

In accordance with (4), we now replace in f the product $[x_i, x_2][x_1, x_j]$ by the difference $[x_i, x_1][x_2, x_j] - x_i[x_1, x_2]x_j$. By the above, we can suppose that

$$f(\bar{x}) = \lambda[x_3, x_4][x_1, x_2] + \sum_{i,j>2} \gamma_{ij}[x_i, x_1][x_2, x_j]x_3 \cdots \hat{x}_i \cdots \hat{x}_i \cdots x_n + \sum_{i,j>2} \delta_{ij} x_i[x_1, x_2]x_j x_3 \cdots \hat{x}_i \cdots \hat{x}_i \cdots x_n,$$

with $\lambda = 0$ if $n \geq 5$. The substitution $x_i \mapsto [x_i, y]$, $x_j \mapsto [x_j, z]v$ transforms our polynomial into

$$\gamma_{ij}[x_i, y][x_j, z]v x_1 \cdots \hat{x}_i \cdots \hat{x}_j \cdots x_n.$$

Consequently, it follows by Lemmas 4(a) and 13 that the coefficients γ_{ij} equal 0 for any i and any j.

If $n = 4$, then

$$f(\bar{x}) = \lambda[x_3, x_4][x_1, x_2] + \delta_{34}x_3[x_1, x_2]x_4 + \delta_{43}x_4[x_1, x_2]x_3.$$

Substituting $x_3 \mapsto [x_3, y]u$ in the identity $f = 0$ and using (1), we arrive at an identity $\delta_{43}x_4[x_1, x_2][x_3, y]u = 0$. And substituting $x_4 \mapsto [x_4, y]u$ in $f = 0$ gives rise to an identity $\delta_{34}x_3[x_1, x_2][x_4, y]u = 0$, which entails $\delta_{34} = \delta_{43} = 0$ in view of Lemmas 4(a) and (13). The case $\lambda \neq 0$ signifies that $[x_3, x_4][x_1, x_2] \in T(\mathcal{V})$. Contradiction.

Now let $n > 4$, i.e., $\lambda = 0$. The substitution $x_3 \mapsto [x_3, y]u$ modulo (1) and (2) transforms $f = 0$ into the following identity:

$$\left(\sum_{i=4}^{n} \delta_{i3}\right)[x_1, x_2][x_3, y]u x_4 \cdots x_n = 0,$$

which, in view of Lemmas 4(a) and 13, means that $\sum_{i=4}^{n} \delta_{i3} = 0$. If in f we replace the coefficient δ_{43} by $-\sum_{i=5}^{n} \delta_{i3}$ and group the terms we arrive at

$$\begin{aligned} f(\bar{x}) &= \sum_{i=5}^{n} \delta_{i3}(x_i[x_1, x_2]x_3x_4 - x_4[x_1, x_2]x_3x_i)x_5 \cdots \hat{x}_i \cdots x_n \\ &\quad + \sum_{j=4}^{n} \delta_{3j}x_3[x_1, x_2]x_j x_4 \cdots \hat{x}_j \cdots x_n \\ &\quad + \sum_{i,j>3}^{n} \delta_{ij}x_i[x_1, x_2]x_j x_3 x_4 \cdots \hat{x}_i \cdots \hat{x}_j \cdots x_n \\ &\overset{(3)}{=} \sum_{i=5}^{n} \delta_{i3}x_3[x_4, x_i][x_1, x_2]x_5 \cdots \hat{x}_i \cdots x_n \\ &\quad + \sum_{j=4}^{n} \delta_{3j}x_3[x_1, x_2]x_j x_4 \cdots \hat{x}_j \cdots x_n \end{aligned}$$

$$+\sum_{i,j>3}^{n} \delta_{ij} x_3 x_i [x_1, x_2] x_j x_4 \cdots \hat{x}_i \cdots \hat{x}_j \cdots x_n$$
$$= x_3 h(x_1, x_2, x_4, \ldots, x_n).$$

By Lemma 4(a), $h \in T(\mathcal{V})$. On the other hand, h has the same form as f, but a smaller number of variables, the latter being a contradiction with the choice of f. The proposition is proved.

4. PROOF OF THE THEOREM: CONSEQUENCES

Proof of the theorem. Lemma 10 and Proposition 4 show that the varieties $\mathcal{TZ}_p$ and $\mathcal{TD}_p$ are almost permutative. Furthermore, every almost permutative variety is contained in $\mathcal{TZ}_p$ or in $\mathcal{TD}_p$, as follows by Props. 2 and 3. At the same time, Proposition 4 implies that such a variety cannot be a proper subvariety of $\mathcal{TZ}_p$ and $\mathcal{TD}_p$. The theorem is proved.

Immediate consequences of Lemma 10 and the theorem are Proposition 1 and Corollary 1.

Proof of Proposition 1. By our theorem, the varieties $\mathcal{TZ}_p$ and $\mathcal{TD}_p$ are almost permutative. Hence any of their nonpermutative algebras cannot generate a proper subvariety, i.e., such generates the entire variety. It remains to appeal to Lemma 10. The proposition is proved.

Proof of Corollary 1. By Zorn's lemma, every nonpermutative variety contains an almost permutative subvariety. Consequently, according to our theorem and Proposition 1, a variety is permutative iff it does not contain algebras $TZ(K_{F,p})$ and $TD(K_{F,p})$.

Below is a statement, which is valid for infinite fields in positive characteristic but is invalid for finite fields and for fields in zero characteristic.

COROLLARY 2. Let F be an infinite field of characteristic $p > 0$. If, in a variety, all nil algebras of index $\leq p + 1$ (for $p > 2$) or of index ≤ 4 (for $p = 2$) are permutative, then the variety is permutative itself.

Proof. In view of Corollary 1, it suffices to verify that neither $TZ(K_{F,p})$ nor $TD(K_{F,p})$ lies in our variety. By Lemma 10, the algebra $TZ(K_{F,p})$ is nonpermutative, and by Lemma 11, it satisfies an identity $x^{p+1} = 0$. Thus $TZ(K_{F,p})$ is a nonpermutative nil algebra of index at most $p + 1$; i.e., $TZ(K_{F,p})$ is not contained in our variety. The algebra $TD(K_{F,p})$ likewise is nonpermutative and satisfies $x^p = 0$ ($p > 2$) or $x^4 = 0$ ($p = 2$). Consequently, $TD(K_{F,p})$, too, is not contained in the variety under consideration.

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